Particle Trajectories in Model Current Sheets,

Part I: Analytical Solutions




by

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Abstract. Approximate analytical solutions are found for two model current sheets. In the first the magnetic field is linear and reverses across a neutral sheet, and the electric field is everywhere uniform, perpendicular to the magnetic field and parallel to the neutral sheet. Charged particles of either sign never come out of the neutral sheet and their energies increase without bound. In the second model a small component of the magnetic field perpendicular to the neutral sheet is added. This component not only serves to bring particles out of the sheet, but turns protons and electrons toward the same direction, $90^{\circ}$ away from the accelerating electric field. The particles are accelerated and then ejected when they have been turned $90^{\circ}$, and the emergent pitch angles to a magnetic line of force will be small if the perpendicular magnetic field component is small.


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Introduction. Ness [1964] has reported the discovery of a magnetically neutral sheet in the earth's geomagnetic tail, with measurements taken on


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board the IMP-1 satellite. A somewhat uniform magnetic field of about $20\ \gamma$ ($1\gamma = 10^{-5}$ gauss $= 10^{-9}$ weber/m$^2$) is found from about 10 $R_e$ (earth-radii) to at least 30 $R_e$ in the anti-solar direction. This field is predominantly in the solar direction above a plane (roughly identified as the magnetic equatorial plane), and in the anti-solar direction below this plane. The magnetic field reverses across a sheet of thickness about 0.1 $R_e$ and goes to zero within this neutral sheet.

Other neutral-, or more generally current-, sheets may occur in other situations, such as a day-side magnetospheric current sheet, neutral sheets occuring in interplanetary space, neutral sheets associated with solar flares, etc.

It is of interest to look at charged particle trajectories about such sheets. Adiabatic theory cannot be used across such a neutral sheet, because the magnetic field changes significantly in distances much less than a gyroradius. The charged particle equations of motion must therefore be either solved analytically, or computed numerically.

This paper is concerned with analytical solutions in two model current sheets. Part II [Speiser, 1965] applies these results to a magnetospheric tail model and also discusses numerical results.

A simple linear model

The simplest model for the fields about a neutral sheet which can be discussed analytically is:

$$\underset{\sim}{B} = - b \frac{x}{d} \hat{e}_y, \qquad (1)$$

[redacted]

$$\underset{\sim}{E} = - a \, \hat{e}_z \tag{2}$$

where b is the strength of the magnetic field when $x = d$, the sheet half-thickness, and a is the strength of the electric field. The physical significance of such an electric field will be discussed in Part II [Speiser, 1965]. This field will merely be assumed for the present treatment.

The co-ordinate system being used is sketched in Figure 1. The above magnetic and electric fields are also indicated in Figure 1.

The equations of motion for a particle in the neutral sheet, using these fields are

$$\ddot{x} = C_1 \, \dot{z} \, x \tag{3}$$

$$\ddot{y} = 0 \tag{4}$$

$$\ddot{z} = - C_3 - C_1 \, x \, \dot{x} \tag{5}$$

where $C_1 = \frac{q}{m} \frac{b}{d}$, and $C_3 = \frac{q}{m} a$. Mks units are used.

Speiser [1964, (a), (b)] has given the solution to these equations for large time. (See Appendix A.) The result is that the particle executes a damped oscillation about $x = 0$ (the amplitude going as $1/t^{1/4}$), while accelerating in the $+\hat{e}_z$ direction for electrons and the $-\hat{e}_z$ direction for protons. This can be understood as follows. The electric field in equation (5) accelerates a proton in the $-\hat{e}_z$

[redacted]

direction. $\dot{z}$ then becomes proportional to -t ignoring the term $-C_1 x\dot{x}$ for the moment. Equation (3) then becomes

$$\ddot{x} = - k x \tag{6}$$

where $k = C_1 \, |\dot{z}|$, or $k = \left(\frac{q}{m}\right)^2 \frac{ab}{d} t$.

Equation (6) is just the equation for oscillatory motion, with spring constant k. k gets larger with time, however, implying that the spring gets stiffer with time, thus the amplitude of oscillation decays.

The term $-C_1 x \dot{x}$ in equation (5) is thus oscillatory and approaches zero, justifying its neglect but the term is not neglected initially in Appendix A. k is proportional to $q^2$, so particles of either sign will oscillate.

The oscillation in x(t) is due to the $\underset{\sim}{V} \times \underset{\sim}{B}$ force which is always toward x = 0 for x either positive or negative, because of the reversal of the magnetic field.

The net result of this simple model is that charged particles of either sign never come out of the neutral sheet, and their energy increases without bound. See Figure 2 for a sketch of the results of this simple model.

<u>The linear model with small perpendicular field added</u>. Consider now the addition of a small magnetic field component perpendicular to the neutral sheet, i.e., in the $+ \hat{e}_x$ direction referring to the co-ordinate system of Figure 1. The magnetic field of equation (1) now becomes:

$$\underset{\sim}{B} = b\left(\eta\ \hat{e}_x - \frac{x}{d}\ \hat{e}_y\right) \tag{7}$$

where $\eta$ will be assumed small.

The equations of motion become:

$$\ddot{x} = C_1\ x\ \dot{z} \tag{8}$$

$$\ddot{y} = C_2\ \eta\ \dot{z} \tag{9}$$

$$\ddot{z} = -\ C_3\ -\ C_1\ x\ \dot{x}\ -\ C_2\ \eta\ \dot{y} \tag{10}$$

where $C_1 = (q/m)\ (b/d)$, $C_2 = (q/m)\ b$, $C_3 = (q/m)a$, $C_1$ and $C_3$ being the same as for the simple model. When $\eta = 0$, the simple model results.

Even without solving these equations, the particle motion can be understood qualitatively as was done for the simple model.

Consider a proton (the arguments also hold for electrons with appropriate changes in sign) incident on this neutral sheet with small velocity. (Strictly speaking, the sheet is not now a neutral sheet.) The proton will be accelerated initially in the negative z direction (see Equation (10)), gaining a velocity, $\dot{z}$, proportional to -t as for the simple model. From Equation (9), there will then be an acceleration in the -y direction proportional to $\dot{z}$ or -t, and thus $\dot{y}$ will be proportional to $-t^2$. (Note that $\ddot{y} \propto C_2\ C_3 \propto (q/m)^2$, so either protons or electrons are turned toward the $-\hat{e}_y$ direction.)

As long as $\dot{z}$ is negative, Equation (8) will imply oscillatory motion in x(t) by the same arguments in the previous section on the simple model. Thus, as long as $\dot{z}$ is negative, the term $-C_1\ x\ \dot{x}$ in

Equation (10) is oscillatory and will be assumed small as a first order approximation. The oscillations may now be either damped or growing depending on whether $\dot{z}$ increases or decreases with time. The last term in Equation (10) grows as $+ t^2$ so that $\dot{z}$ will grow negatively until $\ddot{z}$ goes to zero, and $\dot{z}$ will then diminish in absolute value going to zero and even becoming positive after some time. Thus x(t) will execute damped oscillatory motion until $\ddot{z} = 0$ (the spring gets stiffer). After $\ddot{z}$ becomes positive and until $\dot{z} = 0$, x(t) will execute growing oscillatory motion. After $\dot{z}$ goes positive, however, x(t) will no longer oscillate but will increase exponentially, thus ejecting the particle from the neutral sheet. See Figure 3 for a sketch of the particle trajectories in this model.

Results of the approximate theory. The detailed calculations are contained in Appendix B, to which the following alphabetic equation indices refer. First integrals of the equations of motion (Equations (8), (9) and (10)) are obtained exactly (Equations (i), (j) and (n)). A first approximation is used to obtain the time of ejection, $\tau$, which turns out to be inversely proportional to $\eta$ b (q/m), (Equation (s)). Electrons are therefore ejected from the neutral sheet much sooner than protons. The velocity in the $-\hat{e}_y$ direction at the time of ejection is found to be independent of (q/m), (Equation (w)), and the maximum pitch angle of the emergent particles (Equation (z)) is proportional to $\eta$, and to $|\dot{x}_0/u - c|$, where u is the bulk flow velocity exterior to the neutral sheet, and c is a number which is not very different from 1. This result implies that $\alpha = 0$ for $\eta = 0$, which means that the particles would never come out of the neutral sheet if the small perpendicular field were not added. This agrees with the result of the simple model, and implies that $\alpha$ will be small if $\eta$ is small. Since $\alpha$ is also proportional to $|\dot{x}_0/u - c|$, particles incident on the current sheet with $\dot{x}_0 \approx u$ will all emerge with pitch angles close to zero.

Dungey (1965) suggested that the electric field can be transformed away in this model. Calculations using such a transformation are made in Appendix C. The particle motion in the moving (transformed) system is easily visualized, since only a magnetic field is involved. The perpendicular magnetic field component causes a circular drift in the current sheet while the x-coordinate is oscillating due to the magnetic field reversal ($B_y$). (See, for example Equations (ac), (ak), and (al).) The ejection time is seen to be just a half-period of the circular drift (Equation (am)) and corresponds closely to the approximate result in Appendix B.

The particle energy in the transformed system is a constant, with the initial velocity given by the transformation velocity (in the simple case where the initial velocities are approximately zero in the stationary system). It is therefore apparent that protons and electrons will be ejected with the same velocity, a result which was found before (Appendix B) only for the first and second approximations.

An oscillation frequency about the neutral sheet is found approximately, (Equation (ap)) and the number of oscillations before ejection is determined (Equation (aq)). It is seen that this number is proportional to the square root of the mass, so electrons will execute about 1/40th the number of proton oscillations before ejection.

Summary. Two models of possible current sheets are considered, and approximate analytical results of charged particle trajectories about these sheets are found.

In the simplest model, where the magnetic field varies linearly across a neutral sheet, and there is an electric field perpendicular to the magnetic field and parallel to the current sheet, charged particles of either sign execute damped oscillations about the sheet, accelerating along the sheet. Thus for this simple model, particles never come out of the sheet and their energies go to infinity.

A new model is constructed by adding a small component of the magnetic field perpendicular to the sheet. The addition of this small component not only turns both protons and electrons toward the same direction $90^{\circ}$ away from the accelerating electric field, but serves to eject the particles from the sheet. Protons and electrons are thus accelerated and are then ejected with the same velocity, the electrons being turned much faster and being ejected sooner than the protons.

The pitch angle of ejected particles about a line of force is proportional to the size of the small perpendicular component of magnetic field. Thus, if this component is small, all particles will be ejected nearly along lines of force.

The energy gained is inversely proportional to the square of the perpendicular component of magnetic field. (See Equation (w)). Therefore, in agreement with the simple model, the energies go to infinity when this component is zero. Moreover the energies of the ejected particles can be large if this component is small.

Appendix A. The first integrals of Equations (3) and (5) are:

$$\dot{z} = \dot{z}_0 - C_3 t - \tfrac{1}{2} C_1 \left(x^2 - x_0^2\right) \tag{a}$$

and

$$\tfrac{1}{2}\left(\dot{x}^2 + \dot{z}^2\right) + C_3 z = \tfrac{1}{2}\left(\dot{x}_0^2 + \dot{z}_0^2\right) + C_3 z_0 \tag{b}$$

Equation (b) is just the equation of conservation of energy. The zero subscripted values refer to initial values. Equation (3) becomes, using (a):

$$\ddot{x} = - C_1 x \left(-\dot{z}_0 + \tfrac{1}{2} C_1 \left(x^2 - x_0^2\right) + C_3 t\right) \tag{c}$$

For large enough time, the quantity in parenthesis on the right hand side of Equation (c) will be positive and monotonically increasing implying oscillatory, bounded motion in $x(t)$. Thus for large time Equation (c) is approximately:

$$\ddot{x} \approx - C_1 C_3 x t = -\left(\frac{q}{m}\right)^2 \frac{a\ b}{d} x t \tag{d}$$

The solution to Equation (d) is given by <u>Jahnke and Emde</u> [1945, pp 147] as

$$x = \sqrt{t'} \; Z_{1/3}\left(\tfrac{2}{3} t'^{3/2}\right) \tag{e}$$

where

$$t' = \left(\frac{b\ a}{d}\right)^{1/3} \left(\frac{q}{m}\right)^{2/3} t,$$

and $Z_{1/3}$ is a linear combination of Bessel Functions of the First and Second Kinds, of order one-third. Approximating (e) for large time, [Jahnke and Emde, 1945, pp. 138],

$$x \approx \frac{t^{-1/4}}{\left(\frac{b\ a}{d}\right)^{1/12}\left(\frac{q}{m}\right)^{1/6}} \left[A \cos\left(\frac{2}{3} t^{3/2}\left(\frac{b\ a}{d}\right)^{\frac{1}{2}}\left(\frac{q}{m}\right)\right) + B \sin\left(\frac{2}{3} t^{3/2}\left(\frac{ba}{d}\right)^{\frac{1}{2}}\left(\frac{q}{m}\right)\right)\right] \quad \text{(f)}$$

A and B are constants depending on the initial values.

$z(t)$ may be obtained as a function of time by integrating Equation (a):

$$z(t) \approx -\left(\frac{q}{m}\right)\frac{at^2}{2} + \left(\dot{z}_0 + \left(\frac{q}{m}\right)\left(\frac{b}{d}\right)\frac{x_0^2}{2}\right) t + z_0 \quad \text{(g)}$$

$$+ \text{ smaller oscillatory terms.}$$

From (f), after large time the amplitude of oscillation decays as $1/t^{1/4}$ and from (g) and (b) it is seen that the kinetic energy increases as $t^2$.

Appendix B. First integrals of Equations (8), (9), and (10) are easily obtained. They are:

$$\frac{1}{2}(\dot{x}^2 - \dot{x}_0^2) = -C_3(z - z_0) - \frac{1}{2}(\dot{z}^2 - \dot{z}_0^2) - C_2\,\eta\int_0^t \dot{y}\,\dot{z}\,dt \quad \text{(h)}$$

$$\dot{y} - \dot{y}_0 = C_2\,\eta\,(z - z_0) \quad \text{(i)}$$

$$(\dot{z} - \dot{z}_0) = -C_3 t - \frac{C_1}{2}(x^2 - x_0^2) - C_2\,\eta\,(y - y_0) \quad \text{(j)}$$

Equation (j) becomes, on another integration:

$$(z - z_0) = -\frac{C_3 t^2}{2} - \frac{C_1}{2}\int_0^t x^2 dt - C_2 \eta \int_0^t y dt + C_4 t \tag{k}$$

where

$$C_4 = \dot{z}_0 + \frac{C_1 x_0^2}{2} + C_2 \eta y_0$$

Using Equation (k) with Equation (i), we have:

$$\dot{y} = \dot{y}_0 + \eta\left(-C_5 t^2 + C_6 t - C_7 \int_0^t x^2 dt - C_2^2 \eta \int_0^t y dt\right) \tag{l}$$

where

$$C_5 = \frac{C_2 C_3}{2},\ C_6 = C_2 C_4,\ C_7 = \frac{C_1 C_2}{2}.$$

From Equation (9) we have:

$$C_2 \eta \int_0^t \dot{y}\ \dot{z}\ dt = \frac{1}{2}(\dot{y}^2 - \dot{y}_0^2) \tag{m}$$

So that Equation (h) becomes the energy integral:

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + C_3 z = \frac{1}{2}(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + C_3 z_0 = \frac{E_0}{m} \tag{n}$$

Using Equation (j), Equation (8) becomes:

$$\ddot{x} = -k(t, x, y, \eta)\ x \tag{o}$$

where

$$k(t, x, y, \eta) = C_3 t + \frac{C_1 x^2}{2} + C_2 \eta y + C_8 \tag{p}$$

and

$$C_8 = -\frac{C_1 x_0^2}{2} - C_2 \eta y_0 - \dot{z}_0 = -C_4.$$

k goes to zero when

$$t = -\frac{1}{C_3}\left(\frac{C_1 x^2}{2} + C_2 \eta y + C_8\right) = \tau \qquad (q)$$

$\tau$ is therefore the critical time when k goes to zero and then becomes negative ejecting the particle. ($\dot{z}$ becomes positive.)

Approximations. Integrating Equation (1), we obtain:

$$y = y_0 + \dot{y}_0 t + \frac{C_6 t^2}{2} - \frac{\eta C_5 t^3}{3} \qquad (r)$$

where the integral over $x^2$ in (1) is assumed small since $x(t)$ is oscillating, and the integral over y in (1) is multiplied by $\eta^2$, so will also be neglected in this first approximation. To facilitate finding the critical time of ejection $\tau$, initial conditions are chosen such that $y_0 = \dot{y}_0 = 0$ and $\dot{z}_0 = -C_1 x_c^2/2$, which implies that $C_4 = C_6 = C_8 = 0$. Using Equations (q) and (r) with these initial conditions, $\tau$ becomes:

$$\tau = \frac{\sqrt{6}}{\eta\left(\frac{q}{m}\right) b} \qquad (s)$$

From Equation (r)

$$y(t) = -\frac{\eta C_5 t^3}{3}, \qquad (t)$$

and

$$\dot{y}(t) = -\eta C_5 t^2 \qquad (u)$$

At $t = \tau$, Equations (t) and (u) become:

$$y(\tau) = - \frac{\sqrt{6}\, a}{(\eta b)^2 \left( \frac{q}{m} \right)} \tag{v}$$

and

$$\dot{y}\ (\tau) = - \frac{3a}{\eta b} \tag{w}$$

Thus, in this first approximation, the ejection velocity is independent of (q/m) because electrons are ejected must sooner than protons. (As the next approximation, if y(t) from Equation (t) is used in the integral over y from Equation (k) it is found that $\tau$ is increased by about 30%, $|\dot{y}(\tau)|$ is decreased by about 50%, and the ejection velocity is still independent of (q/m). The first approximation should therefore give the right order of magnitude.)

Pitch angle distributions of emergent particles. In order to calculate the pitch angle of an emergent particle, we must estimate the maximum value $\dot{x}$ can have at the time of ejection. Since x(t) oscillates until $t = \tau$, it is reasonable to assume that $\dot{x}_{max} \approx \dot{x}_0$. Using this assumption and Equation (w) with Equation (7) for the magnetic field with $x = d$, we have:

$$\underset{\sim}{V} = \dot{x}_0\ \hat{e}_x - \frac{3a}{\eta b}\ \hat{e}_y \tag{x}$$

and the cosine of the pitch angle $\alpha$, becomes:

$$\cos\alpha = \frac{(\underset{\sim}{B} \cdot \underset{\sim}{V})}{BV} = 1 - \frac{2\eta^2 \left( \frac{\dot{x}_0}{u} - 3 \right)^2}{36} \tag{y}$$

where $u = a/b$, the bulk flow velocity exterior to the neutral sheet, and only terms of order $\eta^2$ are kept. For small $\alpha$, $\cos\alpha \approx 1 - \alpha^2/2$, so

$$\alpha \approx \frac{\eta \left| \frac{x_0}{u} - 3 \right|}{3} \tag{z}$$

The number 3 in Equation (z) comes from the 3 in Equation (w). This would be decreased to 3/2 by the second approximation mentioned above.

Appendix C. A Lorentz transformation can be made using the fields given by Equations (2) and (7) to a system where the electric field is zero. This is possible for the present model because of the form of the fields and because $B_x$ (Equation (7)) is a constant. If the transformation velocity is chosen as:

$$\underset{\sim}{v} = \frac{-a}{\eta b} \hat{e}_y \tag{aa}$$

then the fields become in the transformed system:

$$\underset{\sim}{E}' = 0$$

$$\underset{\sim}{B}' = b\left(\eta \sqrt{1 - v^2/c^2}\ \hat{e}_{x'} - \frac{x'}{d}\ \hat{e}_{y'}\right) \tag{ab}$$

where the primes refer to the transformed system.

The equations of motion corresponding to Equations (8), (9), and (10) are:

$$\ddot{x}' = C_1\, x'\dot{z}' \tag{ac}$$

$$\ddot{y}' = C_2\eta'\dot{z}' \tag{ad}$$

$$\ddot{z}' = -C_1 x'\dot{x}' - C_2\eta'\dot{y}' \tag{ae}$$

where $\eta' = \eta\sqrt{1 - v^2/c^2}$.

If we borrow the previous result that the x co-ordinate oscillates until $t = \tau$, then the term $-C_1 x'\dot{x}'$ in Equation (ae) will be assumed small initially. Equations (ad) and (ae) then become:

$$\ddot{y}' = \omega\dot{z}' \tag{af}$$

$$\ddot{z}' = -\omega\dot{y}' \tag{ag}$$

where $\omega \equiv C_2\eta'$. These are just the coupled equations for simple harmonic motion, and the solutions are:

$$\dot{y}' = \dot{y}_0' \cos \omega t' + \dot{z}_0' \sin \omega t' \tag{ah}$$

$$\dot{z}' = \dot{z}_0' \cos \omega t' - \dot{y}_0' \sin \omega t' \tag{ai}$$

For transformation velocities small compared to the speed of light, the previous boundary conditions (Appendix B) become in the transformed system

$$\dot{y}_0' = +\frac{a}{\eta b}, \quad \dot{z}_0' = -C_1 x_0^2/2 \tag{aj}$$

For small $\eta$, $\dot{y}_o'$ can be large compared to $\dot{z}_o'$, so Equations (ah) and (ai) are approximately:

$$\dot{y}' = \dot{y}_o' \cos \omega t' \tag{ak}$$

$$\dot{z}' = - \dot{y}_o' \sin \omega t' \tag{al}$$

implying a circular drift in these components. Thus $\dot{z}'$ starts at zero and grows negatively until $\omega t' = \pi/2$, and then decreases in absolute value going to zero at $\omega t' = \pi$. $\dot{z}'$ becomes positive thereafter, and so by the same arguments in the previous section and by Equation (ac) we see that this time is the ejection time.

$$\tau = \frac{\pi}{\omega} = \frac{\pi}{(q/m)\ b\eta} \tag{am}$$

For velocities small compared to the speed of light, this time is about the same in the two systems. The second approximation mentioned in Appendix B replaced the $\sqrt{6}$ in Equation (s) by $\sqrt{10}$ which is very close to $\pi$ from Equation (am). From Equation (ak) at the time of ejection ($\omega t' = \pi$), $\dot{y}'(\tau) = - \dot{y}_o'$. Transforming back to the unprimed system gives

$$\dot{y}(\tau) = - \frac{2a}{\eta b} \tag{an}$$

rather than the factor of 3 (1st approximation Appendix B) or 3/2 (2nd approximation Appendix B).

More information can now be obtained than in Appendix B, by approximating $\dot{z}'$ from Equation (al) as

$$\dot{z}' \approx -\dot{y}_o'/2 \tag{ao}$$

Using Equation (ao) with Equation (ac) we see that the x' co-ordinate will oscillate with frequency

$$\omega_1 = \left(\frac{C_1 \dot{y}_o'}{2}\right)^{1/2} = \left(\frac{(q/m)a}{2\eta d}\right)^{1/2} \tag{ap}$$

and the number of oscillations about the neutral sheet before the particle is ejected is

$$n = \frac{1}{2b}\left(\frac{a}{2(q/m)\eta^3 d}\right)^{1/2} \tag{aq}$$

The above results can be extended to larger velocities (larger $a/\eta b$) by not making the approximation $v \ll c$, and keeping all of the terms from the Lorentz transformation.

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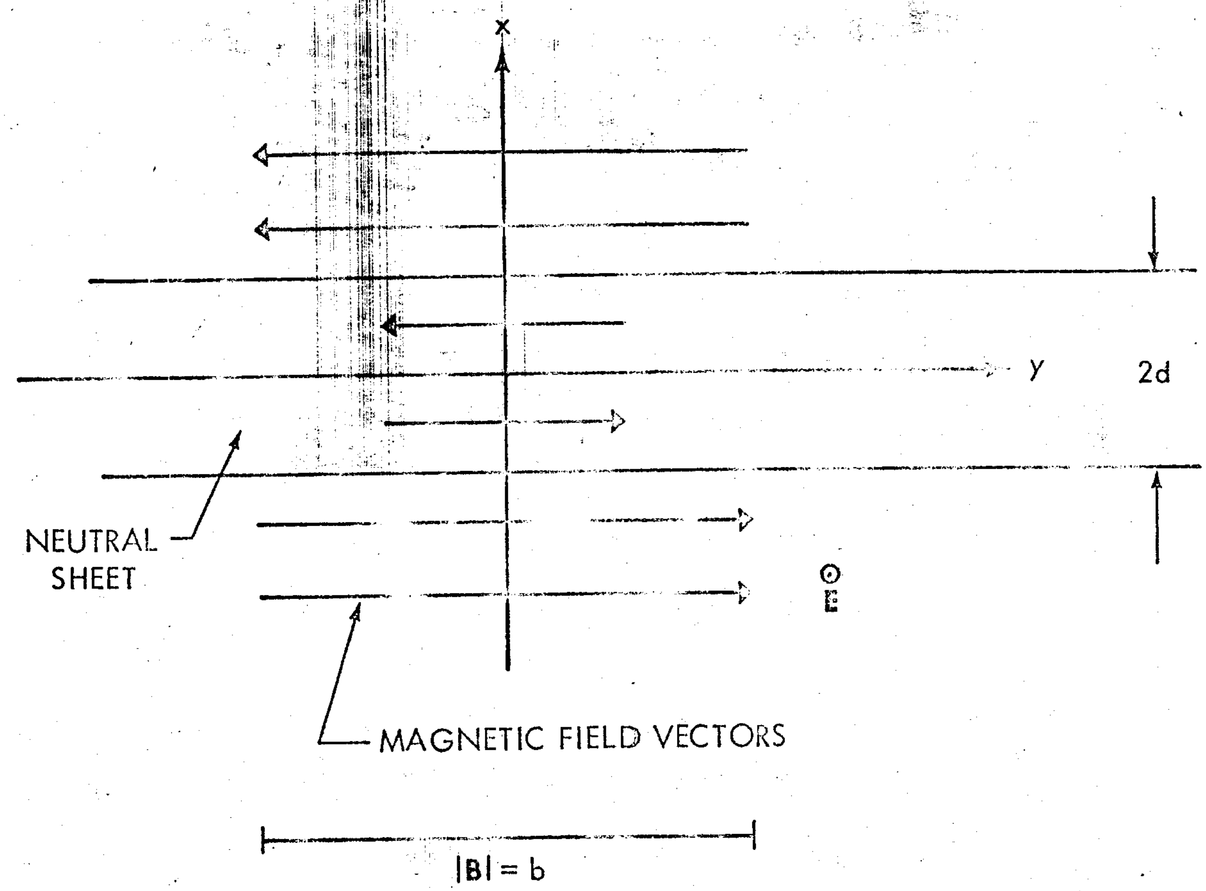
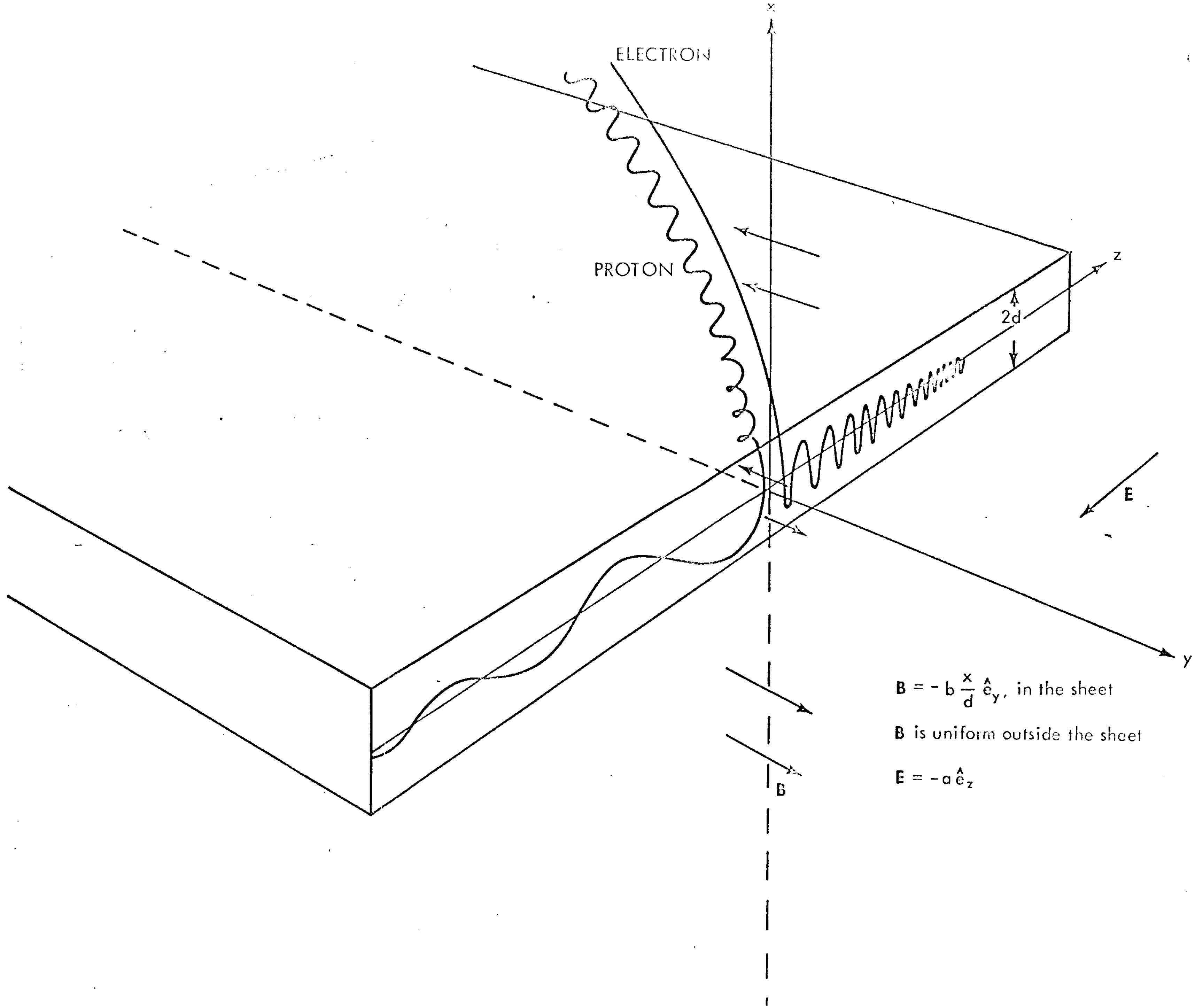
## Figure Captions

1. Simple model.

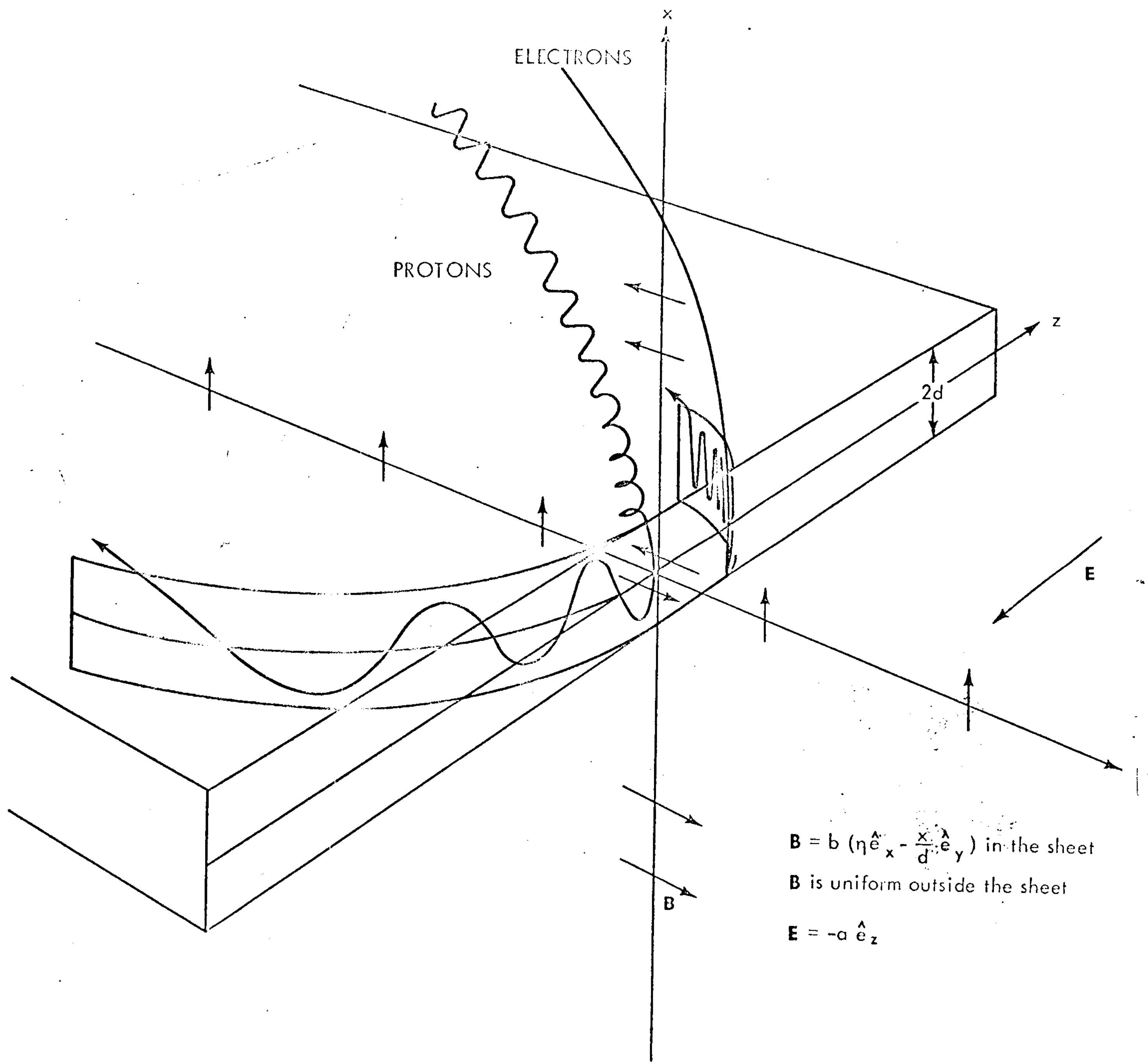
2. Sketch of particle trajectories using the fields of the simple model (linear reversal of the magnetic field across the neutral sheet, no magnetic field component perpendicular to the sheet, and a uniform electric field in the $-\hat{e}_z$ direction). Electrons are accelerated in the $+\hat{e}_z$ direction, protons in the $-\hat{e}_z$ direction. The amplitude of oscillation decays, so particles never come out and their energy goes to infinity.

3. Sketch of particle trajectories using the fields of the simple model plus a small component perpendicular to the neutral sheet. Both protons and electrons oscillate about the sheet accelerating in opposite directions, and are turned toward the same direction by the small magnetic field component perpendicular to the sheet. When the particles are turned $90^\circ$, they are ejected from the neutral sheet. Electrons come out much sooner than protons, hence gain less energy. Electrons also make fewer oscillations than protons before ejection; the above sketch is illustrative and not to scale.


x
y
2d
NEUTRAL SHEET
E
MAGNETIC FIELD VECTORS
|B| = b



x
y
z
ELECTRON
PROTON
2d
E
B
$\mathbf{B} = -b\,\frac{x}{d}\,\hat{e}_y$, in the sheet
$\mathbf{B}$ is uniform outside the sheet
$\mathbf{E} = -a\,\hat{e}_z$


x
ELECTRONS
PROTONS
z
2d
E
y
B
$\mathbf{B} = b\,(\eta \hat{e}_x - \frac{x}{d}\hat{e}_y)$ in the sheet
$\mathbf{B}$ is uniform outside the sheet
$\mathbf{E} = -a\,\hat{e}_z$